Optimization I

Linear Optimization

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Outline

- 1. Organization
- 2. Introduction and Fundamental Terms
- 3. Basic Solution, Optimality Test and Basis Exchange
- 4. The Simplex Algorithm
- 5. Fundamental Theorem of Linear Programming
- 6. 2-Phase-Method
- 7. Duality

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1. Organization

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Organization

Contact

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Organization

Dates

Lecture:

- Monday, from 08:00 10:00 in G 409
- ► Thursday, from 08:00 10:00 in G 209

Tutorials:

- Monday, from 16:00 18:00 in K 107
- ► Friday, from 12:00 14:00 in E 414

Material

Lecture material (e.g. these slides, exercise sheets, \ldots) will be available under

http://uni-ko-ld.de/n5

Organization

Exam

There will be two exams within the next six months:

- Monday, 16.07.2018 in M 001 from 10:00 12:00
- At the beginning of October, date and room are not fixed yet.

You must register in KLIPS in order to participate at the exam. Registration should already be possible from right now.

Register early!

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- 2. Introduction and Fundamental Terms
- 2.1 Graphical Method
- 2.2 Standard Form

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Introduction

Example 2.1 → Handout.

Definition 2.2 (Linear Program in general form) min $c_1 x_1 + \ldots + c_n x_n$ s.t. $a_{i1} x_1 + \ldots + a_{in} x_n = b_i \quad \forall i = 1, \ldots, p$ $a_{i1} x_1 + \ldots + a_{in} x_n \le b_i \quad \forall i = p + 1, \ldots, q$ $a_{i1} x_1 + \ldots + a_{in} x_n \ge b_i \quad \forall i = q + 1, \ldots, m$ $x_j \ge 0 \quad \forall j = 1, \ldots, r$ $x_j \le 0 \quad \forall j = r + 1, \ldots, n$ ► decision variable(s): $x = (x_1, \ldots, x_n)^T$

► feasible region:

$$P := \left\{ \begin{array}{ll} \sum_{j=1}^{n} a_{ij} \, x_j \, = \, b_i & \forall i = 1, \dots, p \\ \sum_{j=1}^{n} a_{ij} \, x_j \, \le \, b_i & \forall i = p+1, \dots, q \\ x \in \mathbb{R}^n : & \sum_{j=1}^{n} a_{ij} \, x_j \, \ge \, b_i & \forall i = q+1, \dots, m \\ & x_j \, \ge \, 0 & \forall j = 1, \dots, r \\ & x_j \, \le \, 0 & \forall j = r+1, \dots, n \end{array} \right\}$$

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Definition 2.3 (Linear Program in general form)

```
 \begin{array}{lll} \min & c_1 x_1 + \ldots + c_n x_n \\ \text{s.t.} & a_{i1} x_1 + \ldots + a_{in} x_n = b_i & \forall i = 1, \ldots, p \\ & a_{i1} x_1 + \ldots + a_{in} x_n \leq b_i & \forall i = p + 1, \ldots, q \\ & a_{i1} x_1 + \ldots + a_{in} x_n \geq b_i & \forall i = q + 1, \ldots, m \\ & x_j \geq 0 & \forall j = 1, \ldots, r \\ & x_j \leq 0 & \forall j = r + 1, \ldots, n \end{array}   (G-LP)
```

• feasible solution: $x \in P$

- objective function value of $x: z(x) := c^{\top}x$
- ▶ optimal solution *x*^{*}: feasible solution with the best objective function value
- ▶ optimal objective function value: objective function value of an optimal solution (if it exists) z* = z(x*) = c[⊤]x*

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Linear programs

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Example 2.4

 $\rightarrow \mathsf{board}$

Observation 2.5

1. The feasible region is a polyhedron (or polytope).

2. There is an optimal solution in a vertex of the polyhedron.

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Notation 2.6

Denote by A_i , the *i*-th row of the matrix $A \in \mathbb{R}^{m \times n}$ and by $A_{\cdot j}$ its *j*-th column.

$$A = \left(A_{\bullet,1}, \dots, A_{\bullet,j}, \dots, A_{\bullet,n}\right) = \begin{pmatrix}A_{1\bullet}\\\vdots\\A_{i\bullet}\\\vdots\\A_{m\bullet}\end{pmatrix} = \begin{pmatrix}a_{11}& \dots & a_{1j}\\\vdots&\ddots&\vdots\\a_{i1}& \dots & a_{ij}\\\vdots&\vdots&\vdots\\a_{m1}& \dots & a_{mj}\\\vdots&\vdots&\vdots\\a_{m1}& \dots & a_{mj}\\\vdots&\dots & a_{mn}\end{pmatrix}$$

Graphical Method

Algorithm 2.7 (Graphical solution method for LPs with two variables)

Input: LP of the form: $\max\{c^{\top}x : Ax \le b; x_j \ge 0, j = 1, 2\}$

- 1. Draw the set of feasible solutions *P* as intersection of the half spaces A_{i} , $x \le b_i$, i = 1, ..., m and $x_j \ge 0, j = 1, 2$
- 2. Choose some $z \in \mathbb{R}$ and draw the line $c^{\top}x = z$
- 3. Find the uniquely determined line $c^\top x = z^*$ parallel to $c^\top x = z$ that satisfies:

3.1 $\exists x^* \in P : c^\top x^* = z^*$ 3.2 z^* is the maximum with this property

Output: Every $x^* \in P$ with $c^\top x^* = z^*$ is an optimal solution of the problem and z^* is the optimal objective function value.

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Infeasibility and Unboundedness

Definition 2.9

An LP is called

- *infeasible* if $P = \emptyset$
- unbounded if the objective function value is unbounded, i. e., there are feasible solutions with arbitrarily good (large/small) objective function values

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Standard Form

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Definition 2.10	(LP in standard form))
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min $c^{\top}x$	
s. t. $Ax = b$	(LP)
$x \ge 0$	

where $c = (c_1, ..., c_n) \in \mathbb{R}^n$ is the cost vector and $A \in \mathbb{R}^{m \times n}$, $m \le n$, with full rank *m* (otherwise: remove redundant rows). $P = \{x \in \mathbb{R}^n : A x = b, x \ge 0\}$ is the feasible region or feasible set.

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Theorem 2.11

Every LP in general form (G-LP) can be transformed into an equivalent LP in standard form (LP).

 $\begin{array}{l} \textbf{Proof.}\\ \rightarrow \text{board} \end{array}$

Linear programs

Example 2.12 (2.4, cont.) \rightarrow board

Idea of an Optimization Algorithm for LP:

- move iteratively from one extreme point (vertex) of the feasible region (polyhedron) to another one
- improve the objective function value in every step
- ► stop if no further improvement is possible

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3. Basic Solution, Optimality Test and Basis Exchange

- 3.1 Basic representation
- 3.2 Optimality condition
- 3.3 Basis exchange

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Definition 3.1 (Basis, Basic Solution) Given a LP in standard form

 $\begin{array}{ll} \min & c^{\top} x \\ \text{s.t.} & A \, x = b \\ & x \geq 0 \end{array}$

A basis of A is a set $\mathcal{B} = \{A_{\cdot B(1)}, \ldots, A_{\cdot B(m)}\}$ of m linearly independent columns of A, where $B = \{B(1), \ldots, B(m)\} \subseteq \{1, \ldots, n\}$ is a subset of the column indices.

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Definition 3.1 (Basis, Basic Solution (cont.)) With respect to *a fixed basis B* we define

- Basic variable x_j with $j \in B$
- Vector of basic variables: $x_B = (x_{B(1)}, \ldots, x_{B(m)})^{\top}$
- Let $N := \{1, ..., n\} \setminus B = \{N(1), ..., N(n-m)\}$
- ▶ Non-basic variable: x_j with $j \in N$
- Vector of non-basic variables: $x_N = (x_{N(1)}, \dots, x_{N(n-m)})^\top$
- ► $A_B := (A_{\cdot B(1)}, \dots, A_{\cdot B(m)}) \in \mathbb{R}^{m \times m}$ nonsingular submatrix of $A_N := (A_{\cdot N(1)}, \dots, A_{\cdot N(n-m)}) \in \mathbb{R}^{m \times (n-m)}$
- ► $c_B := (c_{B(1)}, \dots, c_{B(m)})^\top \in \mathbb{R}^m$ $c_N := (c_{N(1)}, \dots, c_{N(n-m)})^\top \in \mathbb{R}^{n-m}$

Basic representation

```
Ax = b
\iff x_1 A_{\cdot 1} + \ldots + x_n A_{\cdot n} = b
\iff (x_{B(1)}A_{\cdot B(1)} + \cdots + x_{B(m)}A_{\cdot B(m)})
+ (x_{N(1)}A_{\cdot N(1)} + \cdots + x_{N(n-m)}A_{\cdot N(n-m)}) = b
\iff A_B x_B + A_N x_N = b
\iff A_B x_B = b - A_N x_N
```

Since A_B is invertible, we can solve for x_B .

```
\iff x_B = A_B^{-1}b - A_B^{-1}A_N x_N
Basic representation of x w. r. t. B
```

 \longrightarrow Every choice of x_N uniquely determines x_B .

Basic solution

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Definition 3.2

• $x \in \mathbb{R}^n$ is called *basic solution w.r.t. B*, if

 $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} : \ x_N = 0, \ x_B = A_B^{-1}b.$

• A *basic feasible solution (BFS)* is a basic solution with $x_B \ge 0$.

Example 3.3 (Ex. 2.4) \rightarrow board

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Basic solution

Remark 3.4

Any feasible region of an LP is a *polyhedral set*, i.e., it is the intersection of a finite number of halfspaces:

$$\left\{\begin{array}{l} Ax = b \\ x \ge 0 \end{array}\right\} \iff (Ax \le b \land Ax \ge 0 \land x \ge 0)$$

Definition 3.5

An *extreme point* (corner point, vertex) of a polyhedral set $P \subseteq \mathbb{R}^n$ is a point that lies on *n* linearly independent defining hyperplanes of *P*.

Basic solution

Theorem 3.6

Every BFS of (LP) (in standard form) corresponds to an extreme point of the (polyhedral) feasible set $P = \{x \in \mathbb{R}^n : A x = b; x \ge 0\}$ and vice versa.

Proof.

 \rightarrow board Remark 3.7

There may exist more than one basis corresponding to the same BFS or extreme point.

Definition 3.8

A BFS is called *degenerate* if more than n - m variables of the BFS are equal to 0.

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Optimality condition

Theorem 3.9 (Sufficient optimality condition)	
Let x be the basic solution w.r.t. B. If	
$ar{c}^ op:=c^ op-c_B^ op A_B^{-1}A\geq 0,$	
then x is an optimal solution of the LP.	
\overline{c} is called vector of reduced costs.	
Proof.	
\rightarrow board	
Example 3.10	
\rightarrow board	

Optimality condition not fulfilled Assume that the optimality condition is not fulfilled.

I.e. $\bar{c}_{N(s)}^{\top} = c_{N(s)}^{\top} - c_{B}^{\top} A_{B}^{-1} A_{\bullet N(s)} < 0$ for some $N(s) \in N$. Consider the proof of Theorem 3.9:

$$c^{+}x = c_{B}^{+}A_{B}^{-1}b + (c_{N}^{+} - c_{B}^{+}A_{B}^{-1}A_{N})x_{N}$$

Then, $c^{\top}x$ can be improved by increasing $x_{N(s)}$ from 0 to $\delta > 0$ (where $x_{N(j)} = 0 \quad \forall N(j) \in N \setminus \{N(s)\}$)

But the new solution has to be feasible!

 $x_B = A_B^{-1}b - A_B^{-1}A_{\star N(s)} x_{N(s)} \stackrel{!}{\geq} 0$

Setting
$$\widetilde{b} := A_B^{-1} b$$
 and $\widetilde{A}_{\bullet N(s)} := A_B^{-1} A_{\bullet N(s)}$ yields

 $\Longrightarrow x_B = \widetilde{b} - \widetilde{A}_{\star N(s)} x_{N(s)} \ge 0$

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Optimality condition not fulfilled

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Case 1: $\widetilde{a}_{iN(s)} \leq 0 \quad \forall i = 1, \dots, m$

In this case, $x_{N(s)} = \delta$ can be arbitrarily increased without violating the feasibility.

In this way, $c^{\top}x$ can be made arbitrarily small, i. e., the LP is unbounded.

Theorem 3.11 (Criterion for unbounded LPs)

If x is a basic solution and if

 $\overline{c}_{N(s)} < 0$ and $A_B^{-1}A_{\star N(s)} \leq 0$

for some $N(s) \in N$, then the LP is unbounded.

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Case 2: $\exists i \in \{1, \ldots, m\} : \widetilde{a}_{iN(s)} > 0$

$$\begin{aligned} x_{\beta} \stackrel{!}{\geq} 0 & \Longleftrightarrow x_{\beta(i)} = \widetilde{b}_{i} - \widetilde{a}_{iN(s)} \cdot x_{N(s)} \ge 0 \quad \forall i \in \{1, \dots, m\} \\ & \Longrightarrow x_{N(s)} \le \frac{\widetilde{b}_{i}}{\widetilde{a}_{iN(s)}} \quad \forall i \in \{1, \dots, m\} \text{ with } \widetilde{a}_{iN(s)} > 0 \end{aligned}$$

 \rightarrow we increase $x_{N(s)}$ as much as possible i. e., we choose:

$$x_{N(s)} := \min\left\{ rac{\widetilde{b}_i}{\widetilde{a}_{iN(s)}} : \ \widetilde{a}_{iN(s)} > 0
ight\}$$

Case 2: $\exists i \in \{1, ..., m\} : \tilde{a}_{iN(s)} > 0$ (cont.)

Let i = r be an index where this minimum is attained. The *new solution* is then given by:

$$\begin{aligned} x_{N(s)} &= \frac{b_r}{\widetilde{a}_{rN(s)}} \\ x_{N(j)} &= 0 \qquad \forall j \in N \setminus \{N(s)\} \\ x_{B(i)} &= \widetilde{b}_i - \widetilde{a}_{iN(s)} \cdot x_{N(s)} = \widetilde{b}_i - \widetilde{a}_{iN(s)} \frac{\widetilde{b}_r}{\widetilde{a}_{rN(s)}}, \quad i \in \{1, \dots, m\} \\ x_{B(r)} &= 0 \end{aligned}$$

 $\longrightarrow \textbf{Basis exchange:} \begin{cases} x_{B(r)} \text{ leaves the basis} \\ x_{N(s)} \text{ enters the basis} \end{cases}$

Example 3.12 (Ex. 3.2) \rightarrow board

Basis exchange

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Let B be a basis and let $B' := \{B(1), \ldots, B(r-1), N(s), B(r+1), \ldots, B(m)\}$ be the index set after the basis exchange.

 \longrightarrow Is B' again a basis and $x_{B'}$ a basic solution?



Basis exchange

Lemma 3.13 (Steinitz exchange lemma)

Let $B = \{A_1, \ldots, A_m\}$ be a basis of \mathbb{R}^m and let

m

$$A_s = \sum_{i=1}^m \lambda_i A_i \quad (s \notin \{1, \dots, m\})$$

Then it holds that $B' := \{A_1, \dots, A_{r-1}, A_s, A_{r+1}, \dots, A_m\}$ is basis of \mathbb{R}^m iff $\lambda_r \neq 0$.

Proof.

see linear algebra lecture

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Basis exchange

Theorem 3.14

If B is a basis of A, then B' is also a basis of A.

$\begin{array}{l} \textbf{Proof.}\\ \rightarrow \text{board} \end{array}$

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4. The Simplex Algorithm

4.1 Idea

4.2 Simplex tableau

4.3 Algorithm

Idea of the Simplex algorithm

Idea:

As long as the optimality condition is not fulfilled choose some index N(s) with $\overline{c}_{N(s)} < 0$ if the criterion for unboundedness is fulfilled \rightarrow LP is unbounded (Thm. 3.11) else apply the min ratio rule make a basis exchange

Here: Efficient organization of the basis exchange and the optimality test. Let the basis *B* of *A* and the basic feasible solution $x = \begin{pmatrix} x_n \\ x_n \end{pmatrix}$ be given.

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Simplex tableau



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Simplex tableau

Simplex tableau w. r. t. B:

$$T(B) := T_B^{-1}T = \begin{bmatrix} 1 & c^{\top} - c^{\top}_B A_B^{-1}A & -c^{\top}_B A_B^{-1}B \\ 0 & & \\ \vdots & A_B^{-1}A & A_B^{-1}b \\ 0 & & \\ \end{bmatrix}$$
$$= (t_{ij})_{\substack{i=0,\dots,n+1\\j=0,\dots,n+1}} = \begin{bmatrix} 1 & t_{01} & \dots & t_{0n} & t_{0n+1} \\ 0 & t_{11} & \dots & t_{1n} & t_{1n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & t_{m1} & \dots & t_{mn} & t_{mn+1} \end{bmatrix}$$

Since T_B^{-1} is invertible, T(B) represents the same system of linear equations as T.

Interpretation of T(B)

Basic variable: Let $j = B(i) \in B$:

Variable: Let $j = b_i c_j \subset b$. $\blacktriangleright A_B^{-1}A_{\cdot j} = e_i i$ -th unit vector $\triangleright c_j - c_B^{-1}A_{\cdot j}^{-1}A_{\cdot j} = c_j - c_B^{-1}e_i = c_j - c_B(i) = c_j - c_j = 0$ $\Rightarrow T(B)$ contains unit vectors $\begin{pmatrix} 0 \\ e_i \end{pmatrix}$ in columns corresponding basic variable $x_{B(i)} = x_j$

Non-basic variable: Let $j = N(i) \in N^1$:

$$\blacktriangleright A_B^{-1}A_{\cdot j} = A_B^{-1}A_{\cdot N(i)} = \widetilde{A}_{\cdot N(i)} = \begin{pmatrix} a_{1N(i)} \\ \vdots \\ a_{mN(i)} \end{pmatrix}$$

• $c_j - c_B^\top A_B^{-1} A_{\cdot j} = c_{N(i)} - c_B^\top A_B^{-1} A_{\cdot N(i)} = \overline{c}_{N(i)}$ reduced cost of the non-basic variable $x_{N(i)} = x_j$

Vector $A_B^{-1}b$ (last column) • Values of the basic variable $x_B = A_B^{-1}b$

Values of the basic variable
$$x_B = A_B^{-1}$$

simultaneously: $\tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \vdots \end{pmatrix} = A_B^{-1} b$

• simultaneously:
$$b = \left(\begin{array}{c} \vdots \\ \widetilde{b}_{m} \end{array} \right) = A_{B}$$

¹We adapt the notation from Theorem 3.11.

Interpretation of T(B)

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$$\begin{split} & \textbf{Scalar} - c_B^\top A_B^{-1} b: \text{ (``top right'')} \\ & \text{negative objective function value of the current basic solution:} \\ & c^\top x = c_B^\top x_B + c_N^\top x_N = c_B^\top A_B^{-1} b \end{split}$$

Every row in the intial tableau corresponds to one constraint. Known from linear algebra: elementary row operations (like e.g. to multiply T_B^{-1} with T) transform the system of linear equations into an equivalent system of linear equations.

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Simplex tableau

 $\begin{array}{l} \textbf{Definition 4.2} \\ \hline T(B) \text{ is called:} \\ \hline \textbf{feasible} : \Leftrightarrow t_{in+1} \geq 0 \quad \forall i = 1, \dots, m \\ (\Leftrightarrow x_{B(i)} \geq 0 \quad \forall i = 1, \dots, m; \text{ feasibility}) \\ \textbf{optimal} : \Leftrightarrow t_{in+1} \geq 0 \quad \forall i = 1, \dots, m \text{ and } t_{0j} \geq 0 \quad \forall j = 1, \dots, n \\ (\Leftrightarrow \overline{c} \geq 0) \\ \textbf{unbounded} : \Leftrightarrow \exists N(s) \in N: \\ 1. \ t_{0N(s)} = \overline{c}_{N(s)} < 0 \\ 2. \ t_{iN(s)} = \widetilde{a}_{iN(s)} \leq 0 \quad \forall i = 1, \dots, m \end{array}$

Note: These definitions are consistent with the previous definitions of feasibility, the optimality condition and the criterion for unbounded LPs.

Simplex algorithm

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Basis exchange in the tableau:

- ► Choose a non-basic column N(s) with t_{0N(s)} < 0 (negative reduced cost)</p>
- Apply the min ratio rule to this column:

$$\delta := \min\left\{\frac{t_{in+1}}{t_{iN(s)}} : t_{iN(s)} > 0, i \in \{1, \dots, m\}\right\}$$

Let $\delta = \frac{t_{m+1}}{t_{r_{N(s)}}}$ with $r \in \{1, \dots, m\}$

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Definition 4.3

A basis exchange $B' := B \setminus \{B(r)\} \cup \{N(s)\}$ is called *pivot operation*. The column N(s) of T(B) is called *pivot column* and the row *r pivot row*. The entry $t_{rN(s)}$ is called *pivot element*.

Simplex algorithm

Open question: How is the pivot operation in the tableau T(B) realized? How does this lead to T(B')?

• T(B') contains the vector $\begin{pmatrix} 0 \\ e_r \end{pmatrix}$ in column N(s)

• Transform the column N(s) of T(B) into the *r*-th unit vector by applying elementary row transformations.

Simplex algorithm

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Result: $T(B') = (\overline{t}_{ij})_{\substack{i=0,...,m \ j=0,...,m+1}}$ with

- $\blacktriangleright \ \overline{t}_{rj} = \frac{t_{rj}}{t_{rN(s)}} \quad \forall j = 0, \dots, n+1$
- $\blacktriangleright \ \overline{t}_{ij} = t_{ij} \frac{t_{iN(i)}}{t_{rN(i)}} \cdot t_{rj} \quad \forall i = 0, \dots, m, i \neq r \quad \forall j = 0, \dots, n+1$
- $\Rightarrow \overline{t}_{rN(s)} = 1 \text{ and}$
 - $\overline{t}_{iN(s)} = t_{iN(s)} \frac{t_{iN(s)}}{t_{rN(s)}} \cdot t_{rN(s)} = 0$

Algorithm 4.4 (Simplex algorithm) **Input:** LP in standard form: min{ $c^{\top}x : Ax = b, x \ge 0$ } and BFS $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$

w.r.t. to a given basis B. 1: Determine the Simplex tableau T(B)2: while $\exists j \in \{1, \dots, n\}$: $t_{0j} < 0$ do // Tableau not optimal 3: Choose *j* with $t_{0j} < 0$ 4: **if** $t_{ij} \le 0 \ \forall i \in \{1, ..., m\}$ **then**

return "LP is unbounded." 5:

6: else

Find $r \in \{1, \ldots, m\}$ with 7:

$$\frac{t_{r\,n+1}}{t_{r\,j}} = \min\left\{\frac{t_{i\,n+1}}{t_{i\,j}}: t_{i\,j} > 0, i \in \{1, \ldots, m\}\right\}$$

and make a pivot operation with pivot element t_{rj} .

8: return $x^* = \begin{pmatrix} x_0 \\ x_N \end{pmatrix}$ with $x_{\mathcal{B}(i)} = t_{i\,n+1}$ $(i = 1, \ldots, m)$, $x_N = 0$ and objective function value $z^* = c^{\top} x^* = -t_{0\,n+1}$ is an optimal solution of LP.

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Simplex algorithm

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Example 4.5

 $\rightarrow \mathsf{board}$

Simplex algorithm

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Remark 4.6

Open questions:

- 1. Why can we restrict ourselves to basic feasible solutions? \rightarrow Fundamental theorem of linear programming
- 2. How do we find a feasible start basic solution?
- ightarrow 2-phase-method
- 3. Is the simplex algorithm finite?
 - \rightarrow degeneracy, cycling of the simplex, Bland's pivot rule

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5. Fundamental Theorem of Linear Programming

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Fundamental theorem of linear programming

Given: LP in standard form:

min $c^{\top} x$ s.t. $x \in P = \{x \in \mathbb{R}^n : Ax = b\}$ $x \ge 0$

Goal: Show that every feasible LP with bounded objective function value has an *optimal basic solution*.

It follows: Justification that the simplex algorithm restricts to *basic feasible solutions* (i. e., we do not overlook optimal solutions).

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Fundamental theorem of linear programming

Theorem 5.1

If $P := \{x \in \mathbb{R}^n : Ax = b, x \ge 0\} \neq \emptyset$ (i. e. there exists a feasible solution), then there exists a basic feasible solution in P.

Proof.

 $\rightarrow \mathsf{board}$

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Fundamental theorem of linear programming

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Theorem 5.2 (Fundamental theorem of linear programming) If $P := \{x \in \mathbb{R}^n : Ax = b, x \ge 0\} \neq \emptyset$ and if the corresponding LP min $\{c^\top x : Ax = b, x \ge 0\}$ is bounded, then there exists an optimal basic feasible solution.

Proof.	
\rightarrow board	

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6. 2-Phase-Method

6.1 Degenerate simplex iteration

6.2 Cycling

Finding a basic feasible solution

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 ${\bf Goal:}$ Find a feasible start basis (and basic solution) or show the infeasibility of the LP!

Structure of the 2-Phase-Method:

- Phase 1 Determination of a feasible start basis or proof of infeasibility
- Phase 2 (≙ Simplex Algorithm, 4.4) Determination of an optimal basic feasible solution or proof of unboundedness.

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Finding a basic feasible solution

Case (a) original constraints have the form

 $\overline{A}_{i*} x \leq b_i$ with $b_i \geq 0 \quad \forall i = 1, \dots, m$

 \Rightarrow introduce slack variables

A x = b with $A = (\overline{A} \mid I), x = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$

 \Rightarrow select the columns

 $B = \{n+1,\ldots,n+m\}$

corresponding to slack variables as basis columns.

Then, $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ with $x_{B(i)} = x_{n+1} = b_i$ is a b.f.s. (see Example 2.4).

Finding a basic feasible solution

Case (b) $Ax = b, b_i \leq 0, i = 1, ..., m$

Transform the system in such a way, such that as many variables as possible can be identified as slack variables of a constraint
 We introduce *artificial variables* x̂_i for the remaining constraints and obtain:

$$\sum_{j=1}^n a_{ij}x_j + \hat{x}_i = b_i$$

Note that this changes the feasible set of the LP! A solution of it corresponds only to a feasible solution (x₁,..., x_n) of the original LP if all artifical variables equal zero.

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$\begin{array}{l} \textbf{Example 6.1} \\ \rightarrow \text{ board} \end{array}$

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2-Phase-Method - Phase 1

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Idea: Enforce that all artificial variables have the value 0 and pivot them out of the basis.

For this purpose, minimize the *auxiliary objective function*

$$h(\widetilde{x}) := \sum \hat{x}_j$$
 s.t. $\widetilde{A} \, \widetilde{x} = b$, $\widetilde{x} \ge 0$

For this problem, we know a basic feasible solution!

Case 1: $h(\widetilde{x}^*) = \sum \hat{x}_j^* > 0$

Then, there exists no solution for $\widetilde{A} \, \widetilde{x} = b$, $\widetilde{x} \ge 0$ with $\hat{x}_i = 0$ for all artifical variables. Hence, there exists no solution for the original system $A \, x = b$, $x \ge 0$, i.e., the LP is infeasible.

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2-Phase-Method - Phase 1

 $\begin{array}{l} \textbf{Case 2:} \ h(\widetilde{x}^*) = \sum \hat{x}_j^* = 0 \\ \Rightarrow \hat{x}_i^* = 0 \quad \forall i \\ \Rightarrow \text{ there exists a solution of the original system} \\ \textbf{Case 2a):} \ \text{All } \hat{x}_i^* \ \text{are non-basic variables:} \\ \Rightarrow \ \text{all basic variables in the optimal solution of the auxiliary} \\ \text{LP are original variables} \\ \Rightarrow \ \text{a basic feasible solution of the original LP is known} \\ \textbf{Case 2b):} \ \text{There exists a basic variable } \hat{x}_i^* = \hat{x}_{\mathcal{B}(\ell)} = 0: \\ (\text{Note: Such a basic solution is called "degenerate"} \rightarrow \text{see} \\ \text{next chapter}) \\ \text{Pivot the artificial variable out of the basis, i. e., pivot with} \\ t_{\ell_j} \neq 0, \text{ which belongs to an original variable.} \end{array}$

Algorithm 6.2 (2-Phase method)

- **Input:** LP in standard form $\min\{c^{\top}x : Ax = b, x \ge 0\}$ with $b \ge 0$ (multiply A_i , $x = b_i$ by

$$\tilde{a}_{ij} := \begin{cases} a_{ij} & \forall (i,j) : i \in \overline{l}, j \in \{1, \dots, n\} \\ \frac{a_{ij}}{a_{u(i)}} & \forall (i,j) : i \in l, j \in \{1, \dots, n\} \\ 1 & \forall (\overline{l}(k), n+k) : k \in \{1, \dots, |\overline{l}|\} \\ 0 & \text{else} \end{cases}$$
$$\tilde{b}_i := \begin{cases} b_i & \forall i \in \overline{l} \\ \frac{b_{ij}}{a_{u(i)}} & \forall i \in l \end{cases}$$

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// Phase 1

$\tilde{x} := (x_1, \ldots, x_n, \hat{x}_{n+1}, \ldots, \hat{x}_{n+|\tilde{I}|})^\top$

- $B := \{s(i): i \in I\} \cup \{n+1, \dots, n+|\overline{I}|\}, \tilde{x} = \begin{pmatrix} \tilde{x}_B \\ \tilde{x}_N \end{pmatrix} \text{ with } \tilde{x}_B = \tilde{b}$ 6:
- Determine optimal solution \tilde{x}^* of the LP min $\left\{\sum_{i\in \tilde{I}} \tilde{x}_{i+n} : \tilde{A}\tilde{x} = \tilde{b}, \ \tilde{x} \ge 0\right\}$ 7:
- if $\sum_{i \in \overline{I}} \hat{x}_{i+n} > 0$ then return LP is infeasible. 8:
- 9:
- 10: Pivot all artificial variables out of the basis.
- Remove columns n + 1 bis $n + [\overline{n}]$ from the optimal tableau. Replace the objective coefficients of the auxiliary objective by the original objective 11: 12:
- $c^{\top}x.$ 13: Apply elementary row operations to obtain $t_{0B(i)} = 0$ for all basic columns B(i).
- // Start phase 2

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Example 6.3

 $\rightarrow \text{board}$

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Degenerate basic solution/LP

Notes

Definition 6.4

- ► A basic feasible solution *x* is called *degenerate*, if at least one of the basic variables is equal to 0.
- Since $x_N = 0 \Rightarrow |\{i : x_i = 0\}| > n m$ for a degenerate BFS x.
- ► The basis corresponding to a degenerate BFS is also called *degenerate*.
- An LP (in standard form) is called *non-degenerate*, if it is feasible and has no degenerate basic feasible solution.

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Degenerate simplex iteration

Notes

Definition 6.5

A simplex iteration is called *degenerate*, if it does not change the basic solution (i. e., the min ratio rule results in 0) and, for this reason, the objective function value does not change, too.

Example 6.6

 $\rightarrow \mathsf{board}$

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Theorem 6.7

In a non-degenerate LP, the Simplex Method stops after at most $\binom{n}{m}$ iterations (either it finds an optimal solution or it shows unboundedness).

Proof.	
\rightarrow board	

Degenerate simplex iteration

Remark 6.8

- Geometric interpretation: Consider a LP with feasible set
 P = {x ∈ ℝⁿ : A x ≤ b}. If a BFS x̄ is degenerate, then there exist
 more than n inequalities , s.t. A_i, x̄ = b_i, i.e. the number of constraints
 of the LP, which are satisfied with equality is greater than n, e.g. 3
 lines intersect in one point.
- 2. Degenerate bases do not necessarily lead to degenerate simplex iterations.
- 3. Degenerate simplex iterations only occur, if degenerate bases exist.
- 4. Degenerate simplex iterations only become problematic, if a degenerate basis repeats. \rightarrow infinite loop

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Cycling of the Simplex Algorithm

Notes

Notes

Definition 6.9

The simplex algorithm is *cycling*, if one simplex tableau T(B) (w. r. t. to some basis) appears in two different iterations. In this case, the algorithm does not terminate!

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Example 6.10 (Cycling of the Simplex Algorithm, by Beale (1955))

 $\begin{array}{l} \min -\frac{3}{4} x_1 + 20 \, x_2 - \frac{1}{2} \, x_3 + 6 \, x_4 \\ \text{s. t.} \quad \frac{1}{4} \, x_1 - 8 \, x_2 - x_3 + 9 \, x_4 + x_5 &= 0 \\ \frac{1}{2} \, x_1 - 12 \, x_2 - \frac{1}{2} \, x_3 + 3 \, x_4 &+ x_6 &= 0 \\ x_3 &+ x_7 = 1 \\ x_1, \dots, x_7 &\geq 0 \end{array}$

Choose starting basis $B=\{5,6,7\}$ and apply the Simplex-algorithm with the following pivot rule

- Choose the non-basis variable with the smallest reduced costs value $\overline{c}_i < 0.$
- Choose the leaving basis variable according to the min-ratio-rule with the smallest index.

	-3/4	20	-1/2	6	0	0	0	0
	1/4	-8	-1	9	1	0	0	0
	1/2	-12	$-1/_{2}$	3	0	1	0	0
	0	0	1	0	0	0	1	1
	0	4	7/	22	2	0	0	0
	0	-4	- 1/2	33	3	0	0	0
	1	-32	$^{-4}$	36	4	0	0	0
F	0	4	3/2	-15	-2	1	0	0
	0	0	1	0	0	0	1	1
	0	0	-2	18	1	1	0	0
	1	0	8	-84	-12	8	0	0
	0	1	3/8	$-^{15}/_{4}$	-1/2	1/4	0	0
	0	0	1	0	0	0	1	1
	1/4	0	0	-3	-2	3	0	0
	1/8	0	1	$-^{21}/_{2}$	$-^{3}/_{2}$	1	0	0
F	-3/64	1	0	3/16	1/16	-1/8	0	0
	-1/8	0	0	21/2	3/2	-1	1	1

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	-1/2	16	0	0	-1	1	0	0
	-5/2	56	1	0	2	-6	0	0
\longrightarrow	-1/4	16/3	0	1	1/3	$-^{2}/_{3}$	0	0
	5/2	-56	0	0	-2	6	1	1
	-7/4	44	1/2	0	0	-2	0	0
	-5/4	28	1/2	0	1	-3	0	0
\longrightarrow	1/6	$^{-4}$	-1/6	1	0	1/3	0	0
	0	0	1	0	0	0	1	1
	-3/4	20	-1/2	6	0	0	0	0
	1/4	-8	-1	9	1	0	0	0
\longrightarrow	1/2	-12	-1/2	3	0	1	0	0
	0	0	1	0	0	0	1	1

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Cycling

Theorem 6.11 (Bland's pivot rule)

In step (3) of the simplex algorithm, choose

 $j=\min\{j:t_{0j}<0\}$

and in step (5), choose r such that

$$B(r) = \min\left\{B(i): t_{ij} > 0 \text{ and } \frac{t_{in+1}}{t_{ij}} \leq \frac{t_{kn+1}}{t_{kj}} \forall k \text{ with } t_{kj} > 0\right\}.$$

Then the simplex algorithm terminates after finitely many pivot operations (i. e., cycling is prevented).

Example 6.12 (Cycling of the Simplex Algorithm, by Beale (1955))

We apply the Simplex Method with Bland's pivot rule to the problem from Example 6.10.



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	0	2	0	21 2	0	3 2	54	54
	0	0	1	0	0	0	1	1
\rightarrow	0	-2	0	15 2	1	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
	1	-24	0	6	0	2	1	1

An optimal solution of the problem is $x^* = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{3}{4} & 0 & 0 \end{pmatrix}^\top$ with optimal objective value $z = -\frac{5}{4}$.

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Cycling

Notes

Remark 6.13

- 1. If the simplex algorithm is cycling, then all corresponding pivot operations must be degenerate
- 2. Degenerate pivot operations do not necessarily have to lead to cycling.
- 3. Geometric interpretation of cycling: The simplex algorithm gets stuck in one extreme point.

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7. Duality

- 7.1 Dual Problem
- 7.2 Weak Duality
- 7.3 Strong Duality
- 7.4 Complementary slackness conditions
- 7.5 Dual Simplex Algorithm

The	e Dual of a Linear Program	
	Definition 7.1	
	Given a linear program in standard form	
	min $c^{\top}x$	
	s.t. $Ax = b$ (LP)	
	$x \ge 0,$	
	where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.	
	The dual (linear) program or dual problem (DP) of (LP) is defined as:	
	$\max \ b^{\top}\pi$	
	s.t. $A^{\top}\pi \leq c$ (DP)	
	$\pi\gtrless 0$	
	where $\pi = (\pi_1, \ldots, \pi_m)^\top \in \mathbb{R}^m$ are the <i>dual variables</i> and $A^\top \pi \leq c$ are the <i>dual constraints</i> .	
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Dual Problem

Example 7.2 \rightarrow board

Remark 7.3

In the proof of Theorem 3.9 we have already introduced the dual variables $\pi := c_B^\top A_b^{-1}$.

Observation 7.4

If π is an arbitrary solution of the system

 $A^{\top}\pi \leq c,$

then any feasible solution x of (LP) satisfies

$$b^{\top}\pi = (Ax)^{\top}\pi = x^{\top}A^{\top}\pi \leq x^{\top}c = c^{\top}x.$$

Dual Problem

Theorem 7.5	
The dual linear program of (DP) is (LP).	
Proof.	
\rightarrow board	

Dua	al Problem			
	Theorem 7.6			
	An LP in general for	m		
	min	$c^{\top}x$		
	s.t.	A_{i} , $x = b_i$	$\forall i \in M$	
		$A_{i*} x \geq b_i$	$\forall i \in \{1,\ldots,m\} \setminus M =: \overline{M}$	
		$x_j \ge 0$	$\forall j \in L$	
		$x_j \gtrless 0$	$\forall j \in \{1,\ldots,n\} \setminus L =: \overline{L}$	
	has the dual LP			
	max	$b^{ op}\pi$		
	s.t.	$A_{\star j}^{\top}\pi\leq c_j$	$\forall j \in L$	
		$A_{\star j}^{\top}\pi=c_j$	$\forall i \in \overline{L}$	
		$\pi_i \gtrless 0$	$\forall i \in M$	
		$\pi_i \ge 0$	$\forall j \in \overline{M}$	

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Tucker Diagram



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Weak duality theorem

Theorem 7.8 (Weak duality theorem)

Let (LP) be an LP in general form. If x is feasible for (LP) and if π is feasible for (DP), then

 $b^{\top}\pi \leq c^{\top}x$

Proof.

ightarrow board





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Strong duality theorem

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Theorem 7.9 (Strong duality theorem)

Let (LP) and (DP) be a dual pair of linear programs in standard form.

a) If one of the two LPs is unbounded, then the corresponding dual program is infeasible.

- b) If one the two LPs has a finite optimal solution, then so has the other and the optimal objective function values are equal.
- c) (LP) and (DP) may both be infeasible.

Proof. $\rightarrow \mathrm{board}$

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Example 7.10 $\rightarrow \mathsf{board}$

Dual pairs

	solution	function value	solution
finite optimal solution	✓	X	X
feasible solution, unbounded objective function value	X	X	V
no feasible solution	X	\checkmark	\checkmark

Notes

Complementary slackness conditions

Theorem 7.12

Let (LP) be an LP in general form and let (DP) be its dual. Furthermore, let x and π be feasible solutions of (LP) and (DP), respectively.

Then ${\bf x}$ and π are optimal for (LP) and (DP), respectively

$$\iff \begin{cases} u_i := \pi_i \left(A_i, \mathbf{x} - b_i \right) = 0 \qquad \forall i = 1, \dots, m \\ v_j := x_j \left(c_j - A_{\cdot j}^\top \pi \right) = 0 \qquad \forall j = 1, \dots, n \end{cases}$$

Proof. $\rightarrow \mathsf{board}$

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Example 7.14 ightarrow board.

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Example 7.15

If an optimization problem is solved in practice, it may be found that the modeling was inadequate, i.e. that e.g. a constraint was forgotten. It may also happen that a problem needs to be solved for slightly different input data, e.g. for example, the objective function coefficients c may change due to external conditions.

In this case, based on the already known primal or dual optimal solution, an optimal solution of the modified problem can be determined. The advantage is that usually only a few simplex iterations are necessary.

For a given problem min $\{c^{\top}x : Ax \le b, x \ge 0\}$ with primal optimal solution x^* , dual optimal solution π^* and optimal objective function value $c^{\top}x^*$, we consider four cases:

1. The objective function coefficients change: $c \rightsquigarrow c'$

- 2. The right hand side changes: $b \rightsquigarrow b'$
- 3. A new variable $x_{n+1} \ge 0$ is added.
- 4. An additional constraint $A_{m+1} \cdot \leq b_{m+1}$ with $A_{m+1} \cdot \in \mathbb{R}^{1 \times n}, b_{m+1} \in \mathbb{R}$ is added.

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